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On the standing wave for a class of nonlinear Schrödinger equations ☆

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Abstract

This paper is concerned with the standing wave for a class of nonlinear Schrödinger equations

$$i\varphi_t + \Delta\varphi - |x|^2\varphi + \mu|\varphi|^{p-1}\varphi + \gamma|\varphi|^{q-1}\varphi = 0,$$

which describes the attractive Bose–Einstein condensates under a magnetic trap. We establish the existence of the standing wave of the equation. Furthermore, we prove that the standing wave is nonlinearly unstable.

Keywords: Bose–Einstein condensates; Nonlinear Schrödinger equations; Standing wave; Nonlinear instability

1. Introduction

In this paper we investigate a class of nonlinear Schrödinger equations with an isotropic harmonic potential

$$i\varphi_t + \Delta\varphi - |x|^2\varphi + \mu|\varphi|^{p-1}\varphi + \gamma|\varphi|^{q-1}\varphi = 0, \quad \varphi(x,0) = \varphi_0(x), \quad x \in \mathbf{R}^D,\ t \geqslant 0. \tag{1.1}$$

Here $\varphi = \varphi(x,t) : \mathbf{R}^D \times [0,T) \to \mathbf{C}$ is a complex value wave function, where $0 < T \leqslant +\infty$. D is the space dimension, parameters $\mu > 0$, $\gamma > 0$, $i = \sqrt{-1}$, and Δ is the Laplace operator on $\mathbf{R}^D$. $1 < p < q < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 < p < q < \infty$ when $D = 1, 2$. Equation (1.1) models the Bose–Einstein condensates (BEC) with the attractive inter-particle interactions under a magnetic trap [2,8,21,25,27]. The isotropic harmonic potential function $|x|^2$ describes a magnetic field whose role is to confine the movement of particles [2,8,25].

For the nonlinear wave equation as Eq. (1.1), the study of standing wave is regarded as one of the most interesting and important research topics in modern nonlinear wave theory, because many mathematical and physical properties heavily depend on the standing wave (see [3,6,7,9,13,15,16,18,20,24,29,30]).

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In the case of the classic nonlinear Schrödinger equation

$$i\phi_t + \Delta\phi + |\phi|^{p-1}\phi = 0, \quad x \in \mathbf{R}^D,\ t \geqslant 0, \tag{1.2}$$

the standing wave is the solution of the form $\phi(x,t) = e^{iwt}v(x)$, where $w \in \mathbf{R}$ is the frequency and $v(x)$ satisfies

$$-\Delta v + wv - |v|^{p-1}v = 0, \quad v \in H^1(\mathbf{R}^D). \tag{1.3}$$

Here $1 < p < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 < p < +\infty$ when $D = 1, 2$. For Eq. (1.2), Strauss [23] established the existence of the standing wave, and Kwong [14] proved the uniqueness of the standing wave. On the other hand, when $p < 1 + \frac{4}{D}$, Cazenave and Lions [5] proved that the standing wave $e^{iwt}v(x)$ is orbitally stable for any $w > 0$. When $p = 1 + \frac{4}{D}$, Weinstein [26] showed that the standing wave $e^{iwt}v(x)$ is unstable for any $w > 0$. When $p > 1 + \frac{4}{D}$, Berestycki and Cazenave [1] proved that the standing wave $e^{iwt}v(x)$ is unstable for any $w > 0$.

In this paper, we are especially interested in the standing wave of Eq. (1.1). We first establish the existence of the standing wave by constrained variational method. Furthermore, we prove the standing wave of Eq. (1.1) is nonlinearly unstable.

It is deserved to note that Eq. (1.1) is quite different from the classic nonlinear Schrödinger equation (1.2) (also see [31]), which is one of the motivations for us to consider Eq. (1.1). The essential difficulty is that the harmonic potential form $|x|^2\varphi$ and the combined power type nonlinearities $\mu|\varphi|^{p-1}\varphi + \gamma|\varphi|^{q-1}\varphi$ cause to lose the classic scaling invariance, while it exists in the classic nonlinear Schrödinger equation (1.2). Therefore, the classic framework of Grillakis, Shatah and Strauss [11] and the method of Shatah and Strauss [22] to prove the properties of the standing wave do not fit Eq. (1.1). Thus, originating in Goncalves Rebeiro [10], we prove that the standing wave of Eq. (1.1) is nonlinearly unstable.

This paper is organized as follows. In the second section, we give the main results. In the third section, we establish the existence of the standing wave. In the last section, we prove that the standing wave is nonlinearly unstable.

2. Main results

In this section, we shall give the main results. Firstly, we make some preliminaries. We define the natural energy space

$$H := \left\{u \in H^1(\mathbf{R}^D):\ \int |x|^2|u|^2\,dx < \infty\right\}. \tag{2.1}$$

Here and hereafter, for simplicity, we denote $\int_{\mathbf{R}^D} \cdot\, dx$ by $\int \cdot\, dx$. H becomes a Hilbert space, continuously embedded in $H^1(\mathbf{R}^D)$, when endowed with the inner product

$$\langle \varphi, \psi\rangle_H = \int \left[\nabla\varphi\nabla\bar{\psi} + \varphi\bar{\psi} + |x|^2\varphi\bar{\psi}\right] dx, \tag{2.2}$$

whose associated norm we denote by $\|\cdot\|_H$. In addition, we use $\|\cdot\|_2$ to denote the norm of $L^2(\mathbf{R}^D)$ and use $\langle\cdot,\cdot\rangle$ to denote the inner product of $L^2(\mathbf{R}^D)$.

For Eq. (1.1), Oh [19] established the local well-posedness of the Cauchy problem (1.1) in the energy space H as follows (also see Cazenave [4]).

Proposition 2.1. *(See [4,19].) Assume that $1 < p < q < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 < p < q < \infty$ when $D = 1, 2$. For any initial datum $\varphi_0 \in H$. Then there exists a unique solution $\varphi(x,t)$ of the Cauchy problem* (1.1) *in $C([0,T); H)$ for some $T \in [0,\infty)$ (maximal existence time) such that either $T = \infty$ (global existence), or $T < \infty$ and $\lim_{t\to T}\|\varphi\|_H = \infty$ (blow-up).*

Furthermore, for all $t \in [0,T)$, the solution $\varphi(t)$ of the Cauchy problem (1.1) *satisfies the following two conservation laws of the mass*:

$$M(\varphi(t)) := \frac{1}{2}\int |\varphi(t)|^2\,dx = M(\varphi_0), \tag{2.3}$$

and energy

$$E\big(\varphi(t)\big) := \frac{1}{2}\int\Big[\big|\nabla\varphi(t)\big|^2 + |x|^2\big|\varphi(t)\big|^2 - \frac{2}{p+1}\mu\big|\varphi(t)\big|^{p+1} - \frac{2}{q+1}\gamma\big|\varphi(t)\big|^{q+1}\Big]\,dx = E(\varphi_0). \tag{2.4}$$

Proposition 2.2. *(See [4].) Let $\varphi_0 \in H$ and φ be the corresponding solution of the Cauchy problem* (1.1) *on* $[0, T)$*. Put $J(t) = \int |x|^2|\varphi|^2\,dx$. Then one has*

$$\frac{d^2}{dt^2}J(t) = 8\int\Big[|\nabla\varphi|^2 - |x|^2|\varphi|^2 - \frac{D(p-1)}{2(p+1)}\mu|\varphi|^{p+1} - \frac{D(q-1)}{2(q+1)}\gamma|\varphi|^{q+1}\Big]\,dx. \tag{2.5}$$

Lemma 2.1. *(See [28].) Let $1 \leqslant p < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 \leqslant p < \infty$ when $D = 1, 2$. Then the embedding $H \hookrightarrow L^{p+1}(\mathbf{R}^D)$ is compact.*

Lemma 2.2. *(See [26].) Let $\varphi \in H$. Then we have*

$$\int |\varphi|^2\,dx \leqslant \frac{2}{D}\Big(\int |\nabla\varphi|^2\,dx\Big)^{1/2}\Big(\int |x|^2|\varphi|^2\,dx\Big)^{1/2}. \tag{2.6}$$

For Eq. (1.1), the standing wave is the solution with the form $\varphi(x, t) = e^{iwt}u(x)$, where $w \in \mathbf{R}$ is the frequency and $u(x)$ is the ground state solution of the following nonlinear elliptic equation:

$$-\Delta u + |x|^2 u + wu - \mu|u|^{p-1}u - \gamma|u|^{q-1}u = 0, \quad u \in H. \tag{2.7}$$

The natural definition of stability of standing wave is orbital stability (also be called nonlinear stability) [17].

Definition 2.1. The standing wave $e^{iwt}u(x)$ is called orbitally stable if for any $\varepsilon > 0$ there exists $\sigma > 0$ such that for any φ_0 satisfying

$$\inf_{\theta\in\mathbf{R}}\big\|\varphi_0 - e^{i\theta}u(x)\big\|_H < \sigma,$$

Eq. (1.1) has a corresponding global solution $\varphi(t) \in H$ with $\varphi(0) = \varphi_0$ satisfying

$$\inf_{\theta\in\mathbf{R}}\big\|\varphi(t) - e^{i\theta}u(x)\big\|_H < \varepsilon \quad \text{for all } t > 0.$$

Otherwise, the standing wave $e^{iwt}u(x)$ is nonlinearly unstable.

In this paper, we first establish the existence of the standing wave by constrained variational method, which originates in Strauss [23]. Then we prove the standing wave of Eq. (1.1) is nonlinearly unstable, which originates in Goncalves Rebeiro [10]. The main results read as follows.

Theorem 2.1 *(Existence). Let $1 < p < q < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 < p < q < \infty$ when $D = 1, 2$. For any $w > 0$, the nontrivial standing wave $e^{iwt}u(x)$ of Eq.* (1.1) *exists.*

Theorem 2.2 *(Instability). Let $1 + \frac{4}{D} \leqslant p < q < \frac{D+2}{D-2}$ when $D \geqslant 3$ and $1 + \frac{4}{D} \leqslant p < q < \infty$ when $D = 1, 2$. If*

$$\int\Big[4|\nabla u|^2 - \frac{\mu D(p-1)[D(p-1)+4]}{4(p+1)}|u|^{p+1} - \frac{\gamma D(q-1)[D(q-1)+4]}{4(q+1)}|u|^{q+1}\Big]\,dx < 0,$$

then, for any $w > 0$, the standing wave of Eq. (1.1) *is nonlinearly unstable.*

3. Existence of the standing wave

In this section, we shall establish the existence of the standing wave. Firstly, we define some functionals in the energy space H,

$$L(u)=\frac{1}{2}\int\left[|\nabla u|^2+|x|^2|u|^2+w|u|^2-\frac{2}{p+1}\mu|u|^{p+1}-\frac{2}{q+1}\gamma|u|^{q+1}\right]dx, \tag{3.1}$$

$$I(u)=\int\left[|\nabla u|^2+|x|^2|u|^2+w|u|^2-\mu|u|^{p+1}-\gamma|u|^{q+1}\right]dx \tag{3.2}$$

and

$$P(u)=\int\left[|\nabla u|^2-|x|^2|u|^2-\frac{D(p-1)}{2(p+1)}\mu|u|^{p+1}-\frac{D(q-1)}{2(q+1)}\gamma|u|^{q+1}\right]dx. \tag{3.3}$$

Then pose a variational problem

$$\begin{cases} d=\inf\limits_{u\in\Omega}L(u),\\ \Omega:=\left\{u\in H\setminus\{0\}\colon\ I(u)=0\right\}. \end{cases} \tag{3.4}$$

Lemma 3.1. *Let $1<p<q<\frac{D+2}{D-2}$ when $D\geqslant 3$ and $1<p<q<\infty$ when $D=1,2$. For any $w>0$, there must exist $u\in\Omega$ such that*

$$d=L(u)=\min_{v\in\Omega}L(v).$$

Furthermore, $d>0$.

Proof. Choose a minimizing sequence $\{u_n\}_{n\in\mathbf{N}}\subset H\setminus\{0\}$ of the variational problem (3.4). Therefore,

$$\begin{aligned} d&=\lim_{n\to+\infty}L(u_n)\\ &=\lim_{n\to+\infty}\frac{1}{2}\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2-\frac{2}{p+1}\mu|u_n|^{p+1}-\frac{2}{q+1}\gamma|u_n|^{q+1}\right]dx \end{aligned} \tag{3.5}$$

and

$$I(u_n)=\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2-\mu|u_n|^{p+1}-\gamma|u_n|^{q+1}\right]dx=0. \tag{3.6}$$

Since (3.6), Sobolev inequality, Lemma 2.2, Cauchy inequality, $w>0$ and $1<p<q$, it follows that

$$\begin{aligned} &\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2\right]dx\\ &\quad=\mu\int|u_n|^{p+1}dx+\gamma\int|u_n|^{q+1}dx\\ &\quad\leqslant C\left(\left[\int\left[|\nabla u_n|^2+|u_n|^2\right]dx\right]^{\frac{p+1}{2}}+\left[\int\left[|\nabla u_n|^2+|u_n|^2\right]dx\right]^{\frac{q+1}{2}}\right)\\ &\quad\leqslant C\left(\left[\int\left[|\nabla u_n|^2+|x|^2|u_n|^2\right]dx\right]^{\frac{p+1}{2}}+\left[\int\left[|\nabla u_n|^2+|x|^2|u_n|^2\right]dx\right]^{\frac{q+1}{2}}\right)\\ &\quad\leqslant C\left(\left[\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2\right]dx\right]^{\frac{p+1}{2}}+\left[\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2\right]dx\right]^{\frac{q+1}{2}}\right). \end{aligned} \tag{3.7}$$

Here and hereafter, we use C to denote various positive constants. Therefore there exists a positive constant C such that

$$\int\left[|\nabla u_n|^2+|x|^2|u_n|^2+w|u_n|^2\right]dx\geqslant C>0,$$

which then implies from (3.6) that

$$\int |u_n|^{p+1}\,dx \geqslant C > 0, \qquad \int |u_n|^{q+1}\,dx \geqslant C > 0. \tag{3.8}$$

Substituting (3.6) into $L(u_n)$, we get

$$L(u_n) = \frac{\mu(p-1)}{2(p+1)} \int |u_n|^{p+1}\,dx + \frac{\gamma(q-1)}{2(q+1)} \int |u_n|^{q+1}\,dx. \tag{3.9}$$

Then it follows from (3.8) and (3.9) that $d > 0$.

On the other hand, it follows from (3.5), (3.9) and (3.6) that $\{u_n\}_{n\in\mathbf{N}}$ are bounded in H. Then there must exist $u \in H$ such that the subsequence of $\{u_n\}_{n\in\mathbf{N}} \subset H \setminus \{0\}$, which we still denoted by $\{u_n\}_{n\in\mathbf{N}}$, satisfy

$$u_n \rightharpoonup u \quad \text{as } n \to +\infty. \tag{3.10}$$

Therefore, from Lemma 2.1, we know that

$$\begin{aligned} &u_n \to u \quad \text{in } L^2(\mathbf{R}^D),\\ &u_n \to u \quad \text{in } L^{p+1}(\mathbf{R}^D),\\ &u_n \to u \quad \text{in } L^{q+1}(\mathbf{R}^D). \end{aligned} \tag{3.11}$$

So, it follows from (3.10), (3.11) and $\int[|\nabla u_n|^2 + |x|^2|u_n|^2]\,dx$ being coercive and convex that

$$I(u) \leqslant \lim_{n\to+\infty} \inf I(u_n) \leqslant \lim_{n\to+\infty} I(u_n) = 0. \tag{3.12}$$

Then there exists a real number $\beta \in (0, 1]$ such that $I(\beta u) = 0$. Note the fact that $u \neq 0$ since (3.9), (3.11) and $d > 0$. Thus, we have $\beta u \in \Omega$, which implies from (3.4) that $L(\beta u) \geqslant d$. Then from $\beta \in (0, 1]$, $1 < p < q$ and $\beta u \in \Omega$, we obtain

$$\begin{aligned} &\beta^{p+1}\left[\int \frac{\mu(p-1)}{2(p+1)}|u|^{p+1}\,dx + \int \frac{\gamma(q-1)}{2(q+1)}|u|^{q+1}\,dx\right]\\ &\quad\geqslant \int \frac{\mu(p-1)}{2(p+1)}\beta^{p+1}|u|^{p+1}\,dx + \int \frac{\gamma(q-1)}{2(q+1)}\beta^{q+1}|u|^{q+1}\,dx\\ &\quad= L(\beta u)\\ &\quad\geqslant d. \end{aligned} \tag{3.13}$$

At the same time, it follows from (3.5), (3.9)–(3.11) that

$$\frac{1}{2}\int\left[\frac{\mu(p-1)}{p+1}|u|^{p+1} + \frac{\gamma(q-1)}{q+1}|u|^{q+1}\right]dx = \lim_{n\to+\infty}\frac{1}{2}\int\left[\frac{\mu(p-1)}{p+1}|u_n|^{p+1} + \frac{\gamma(q-1)}{q+1}|u_n|^{q+1}\right]dx = d. \tag{3.14}$$

Therefore, it follows from (3.13), (3.14) and $\beta \in (0, 1]$ that $\beta = 1$. Then $u \in \Omega$ and $L(u) = d$. The proof is completed. □

Proof of Theorem 2.1. Let Γ be the set of the minimizers for the variational problem (3.4). From Lemma 3.1, there exists $u \in \Gamma$. Then there exists a Lagrange multiplier $\Lambda \in \mathbf{R}$ such that

$$L' + \Lambda I' = 0. \tag{3.15}$$

Here L' and I' denote their Fréchet derivatives, respectively. Therefore, we get

$$\begin{aligned} &-\Delta u + |x|^2 u + wu - \mu|u|^{p-1}u - \gamma|u|^{q-1}u\\ &\quad + \Lambda\left[-2\Delta u + 2|x|^2 u + 2wu - (p+1)\mu|u|^{p-1}u - (q+1)\gamma|u|^{q-1}u\right] = 0. \end{aligned} \tag{3.16}$$

Multiplying (3.16) with u and integrating on $\mathbf{R}^D$, using $I(u) = 0$, we get $\Lambda \equiv 0$. Therefore, nontrivial u satisfies the nonlinear elliptic equation (2.7)

$$-\Delta u + |x|^2 u + wu - \mu|u|^{p-1}u - \gamma|u|^{q-1}u = 0, \quad u \in H.$$

Therefore, for any $w > 0$, the nontrivial standing wave $e^{iwt}u(x)$ of Eq. (1.1) exists. The proof is complete. □

4. Instability of the standing wave

In this section, we shall prove that the standing wave $e^{iwt}u(x)$ of Eq. (1.1) is nonlinearly unstable.

From Lemma 3.1, we know the variational problem (3.4) is attained. For any $w > 0$, let u be the minimizer of the variational problem (3.4). Then $L(u) = d$. At the same time, it follows from the proof of Theorem 2.1 that u is the solution of the nonlinear elliptic equation (2.7). Therefore, $e^{iwt}u(x)$ is a standing wave. Also since Lemma 3.1, u is called a ground state solution of Eq. (2.7).

Definition 4.1. For $\varepsilon_0 > 0$, we define a tabular neighborhood around the orbit $\{e^{i\theta}u\colon \theta \in \mathbf{R}\}$ by

$$U_{\varepsilon_0}(u) = \Big\{\xi \in H\colon \inf_{\theta\in\mathbf{R}} \big\|\xi - e^{i\theta}u\big\|_H < \varepsilon_0\Big\}.$$

Definition 4.2. For any $\varphi_0 \in U_{\varepsilon_1}(u)$, we define the maximal existence time from $U_{\varepsilon_1}(u)$ as follows

$$T(\varphi_0) = \sup\big\{T > 0\colon \varphi(t) \in U_{\varepsilon_1}(u),\ 0 \leqslant t < T\big\},$$

where $\varphi(t)$ is the solution of Eq. (1.1) corresponding with the initial datum φ_0.

In the following, we define a set

$$\Pi = \big\{\xi \in U_{\varepsilon_1}(u)\colon E(\xi) < E(u),\ \|\xi\|_2 = \|u\|_2,\ P(\xi) < 0\big\}. \tag{4.1}$$

Here $E(\xi)$ is the energy functional as (2.4), and $P(\xi)$ is defined as (3.3).

Lemma 4.1. *If $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1} < 0$, then there exist $\varepsilon_0 > 0$, $\sigma_0 > 0$ and a mapping*

$$\lambda : U_{\varepsilon_0}(u) \to (1-\sigma_0, 1+\sigma_0)$$

such that

$$I\big(\xi^\lambda\big) = 0, \quad \textit{for all } \xi \in U_{\varepsilon_0}(u).$$

Here $u^\lambda = \lambda^{D/2}u(\lambda x)$ and $\xi^\lambda = \lambda^{D/2}\xi(\lambda x)$ for any $\lambda > 0$.

Proof. After simple calculations, we have

$$\frac{\partial}{\partial\lambda} I\big(\xi^\lambda\big)\Big|_{\lambda=1,\,\xi=u} = \left\langle I'(u), \frac{\partial u^\lambda}{\partial\lambda}\Big|_{\lambda=1}\right\rangle.$$

We can affirm that $\langle I'(u), \frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}\rangle \neq 0$. Otherwise, $\langle I'(u), \frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}\rangle = 0$. Then $\frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}$ would be the tangent to the set Ω at $u(x)$, where Ω is defined as (3.4). Therefore, in the case, $\langle L''(u)\frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}, \frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}\rangle \geqslant 0$ because $u(x)$ is the minimizer of the variational problem (3.4) from Lemma 3.1. This implies a contradiction of the assumption

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2} E\big(u^\lambda\big)\Big|_{\lambda=1} &= \frac{\partial^2}{\partial\lambda^2} L\big(u^\lambda\big)\Big|_{\lambda=1}\\ &= \left\langle L''(u)\frac{\partial u^\lambda}{\partial\lambda}\Big|_{\lambda=1}, \frac{\partial u^\lambda}{\partial\lambda}\Big|_{\lambda=1}\right\rangle\\ &< 0.\end{aligned}$$

Therefore, $\frac{\partial}{\partial\lambda}I(\xi^\lambda)|_{\lambda=1,\,\xi=u} = \langle I'(u), \frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}\rangle \neq 0$.

Combined this result with $I(\xi^\lambda)|_{\lambda=1,\,\xi=u} = I(u) = 0$, it follows from the implicit function theorem that the result is true. $\square$

Lemma 4.2. *If $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1} < 0$, then there exist $\varepsilon_1 > 0$, $\sigma_1 > 0$ such that for any $\xi \in U_{\varepsilon_1}(u)$, satisfying $\|\xi\|_2 = \|u\|_2$, there exists $\lambda \in (1-\sigma_1, 1+\sigma_1)$ such that*

$$E(u) < E(\xi) + (\lambda - 1)P(\xi).$$

Proof. Since $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1}<0$ and $\frac{\partial^2}{\partial\lambda^2}E(\xi^\lambda)$ is continuous with respect to λ and ξ, there exist $\varepsilon_1>0$ and $\sigma_1>0$ such that

$$\frac{\partial^2}{\partial\lambda^2}E\big(\xi^\lambda\big)<0,\quad \text{for any } \lambda\in(1-\sigma_1,1+\sigma_1) \text{ and } \xi\in U_{\varepsilon_1}(u). \tag{4.2}$$

After simple calculation, we have

$$\frac{\partial}{\partial\lambda}E\big(\xi^\lambda\big)\Big|_{\lambda=1}=P(\xi). \tag{4.3}$$

At the same time, the Taylor expansion of $E(\xi^\lambda)$ at $\lambda=1$ reads that for any $\xi\in U_{\varepsilon_1}(u)$,

$$E\big(\xi^\lambda\big)=E\big(\xi^\lambda\big)\big|_{\lambda=1}+\frac{\partial}{\partial\lambda}E\big(\xi^\lambda\big)\Big|_{\lambda=1}\cdot(\lambda-1)+\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}E\big(\xi^\lambda\big)\Big|_{\lambda\in(1-\sigma_1,1+\sigma_1)}\cdot(\lambda-1)^2.$$

Then it follows from (4.2) and (4.3) that

$$E\big(\xi^\lambda\big)<E(\xi)+P(\xi)(\lambda-1),\quad \text{for any } \lambda\in(1-\sigma_1,1+\sigma_1) \text{ and } \xi\in U_{\varepsilon_1}(u). \tag{4.4}$$

From Lemma 4.1, we can take $0<\varepsilon_1<\varepsilon_0$ and $0<\sigma_1<\sigma_0$ such that

$$I\big(\xi^\lambda\big)=0,\quad \text{for any } \lambda\in(1-\sigma_1,1+\sigma_1) \text{ and } \xi\in U_{\varepsilon_1}(u). \tag{4.5}$$

In other words, there exist $\varepsilon_1>0$ and $\sigma_1>0$ such that $\xi^\lambda\in\Omega$, which implies from Lemma 3.1 that

$$L\big(\xi^\lambda\big)\geqslant L(u),\quad \text{for any } \lambda\in(1-\sigma_1,1+\sigma_1) \text{ and } \xi\in U_{\varepsilon_1}(u). \tag{4.6}$$

On the other hand, for any $\xi\in U_{\varepsilon_1}(u)$ satisfying $\|\xi\|_2=\|u\|_2$, we have

$$\big\|\xi^\lambda\big\|_2=\|\xi\|_2=\|u\|_2. \tag{4.7}$$

Therefore, it follows from (4.6) and (4.7) that

$$\begin{aligned}E\big(\xi^\lambda\big)&=L\big(\xi^\lambda\big)-wM\big(\xi^\lambda\big)\\&\geqslant L(u)-wM\big(\xi^\lambda\big)\\&=L(u)-wM(\xi)\\&=L(u)-wM(u)\\&=E(u).\end{aligned} \tag{4.8}$$

Here $M(u)$ is the mass functional as (2.3). Thus by (4.4) and (4.8), the proof is completed. $\square$

Lemma 4.3. *If $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1}<0$, then for any $\varphi_0\in\Pi$, there exists $\sigma_0=\sigma(\varphi_0)>0$ such that the solution $\varphi(t)$ of Eq.* (1.1) *corresponding with the initial datum φ_0 satisfies*

$$P\big(\varphi(t)\big)<-\sigma_0,\quad \textit{for } 0\leqslant t<T(\varphi_0).$$

Proof. Let $\varphi_0\in\Pi$, which reads that $E(\varphi_0)<E(u)$, $\|\varphi_0\|_2=\|u\|_2$ and $P(\varphi_0)<0$. Put

$$\sigma_2:=E(u)-E(\varphi_0)>0.$$

It follows from Lemma 4.2 that

$$P\big(\varphi(t)\big)(\lambda-1)+E\big(\varphi(t)\big)>E(u), \tag{4.9}$$

for any $\lambda\in(1-\sigma_1,1+\sigma_1)$ and $0\leqslant t<T(\varphi_0)$. Then by the energy conservation (2.4), we get

$$P\big(\varphi(t)\big)(\lambda-1)>E(u)-E\big(\varphi(t)\big)=E(u)-E(\varphi_0)=\sigma_2>0. \tag{4.10}$$

Thus,

$$P\big(\varphi(t)\big)\neq 0,\quad \text{for any } \lambda\in(1-\sigma_1,1+\sigma_1) \text{ and } 0\leqslant t<T(\varphi_0). \tag{4.11}$$

At the same time, noting the fact $P(\varphi_0) < 0$ and that $P(\varphi(t))$ is continuous with respect to t, therefore, we have

$$P\big(\varphi(t)\big) < 0, \quad \text{for any } 0 \leqslant t < T(\varphi_0). \tag{4.12}$$

So, it follows from (4.10) and (4.12) that

$$-\sigma_1 < \lambda - 1 < 0, \quad \text{for any } 0 \leqslant t < T(\varphi_0).$$

Then

$$P\big(\varphi(t)\big) < -\frac{\sigma_2}{\sigma_1} := -\sigma_0 < 0, \quad \text{for any } 0 \leqslant t < T(\varphi_0).$$

The proof is completed. □

Proof of Theorem 2.2. Since u is the ground state solution of Eq. (2.7), it follows from the Pohozaev identities of Eq. (2.7) that

$$P(u) = \int \left[|\nabla u|^2 - |x|^2|u|^2 - \frac{D(p-1)}{2(p+1)}\mu|u|^{p+1} - \frac{D(q-1)}{2(q+1)}\gamma|u|^{q+1} \right] dx = 0. \tag{4.13}$$

Therefore it follows from (4.13) that

$$\begin{aligned}
\frac{\partial^2}{\partial\lambda^2} E\big(u^\lambda\big)\Big|_{\lambda=1} &= \int \left[|\nabla u|^2 + 3|x|^2|u|^2 - \frac{\mu D(p-1)[D(p-1)-2]}{4(p+1)}|u|^{p+1} \right. \\
&\qquad \left. - \frac{\gamma D(q-1)[D(q-1)-2]}{4(q+1)}|u|^{q+1} \right] dx \\
&= \int \left[4|\nabla u|^2 - \frac{\mu D(p-1)[D(p-1)+4]}{4(p+1)}|u|^{p+1} - \frac{\gamma D(q-1)[D(q-1)+4]}{4(q+1)}|u|^{q+1} \right] dx.
\end{aligned}$$

If $\int [4|\nabla u|^2 - \frac{\mu D(p-1)[D(p-1)+4]}{4(p+1)}|u|^{p+1} - \frac{\gamma D(q-1)[D(q-1)+4]}{4(q+1)}|u|^{q+1}]\,dx < 0$, we have $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1} < 0$.

In addition, after simple calculation, we have $\frac{\partial}{\partial\lambda}E(u^\lambda)|_{\lambda=1} = P(u)$ and $\frac{\partial}{\partial\lambda}E(u^\lambda) = \lambda^{-1}P(u^\lambda)$. Thus combining these results with $\frac{\partial^2}{\partial\lambda^2}E(u^\lambda)|_{\lambda=1} < 0$, we get

$$E\big(u^\lambda\big) < E(u), \quad P\big(u^\lambda\big) < P(u) = 0, \quad \text{as } \lambda > 1.$$

Furthermore, noting the fact that

$$\big\|u^\lambda\big\|_2 = \|u\|_2,$$

we get that

$$u^\lambda \in \Pi, \quad \text{as } \lambda > 1.$$

Now we take the initial datum

$$\varphi_0 = u^\lambda, \quad \text{as } \lambda > 1 \text{ and } \lambda \to 1.$$

Obviously,

$$\|\varphi_0 - u\|_H \to 0 \quad \text{and} \quad \varphi_0 \in \Pi.$$

By Lemma 4.3, there exists $\sigma_0 = \sigma(\varphi_0) > 0$ such that the solution $\varphi(t)$ of Eq. (1.1) corresponding with the initial datum φ_0 satisfies

$$P\big(\varphi(t)\big) < -\sigma_0, \quad \text{for } 0 \leqslant t < T(\varphi_0).$$

Then it follows from Proposition 2.2 that $\frac{d^2}{dt^2}J(t) = 8P(\varphi(t)) < -8\sigma_0$. Therefore, from Lemma 2.2, using the method of Glassey [12], the solution $\varphi(t)$ blows up in a finite time. The proof is completed. □

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